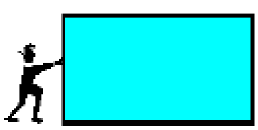
The Area Model of Multiplication Across the Curriculum in Grades 2-12



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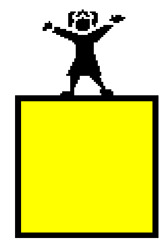

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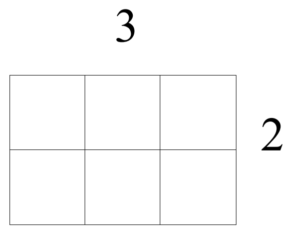
The Area Model of Multiplication

Every multiplication fact can be illustrated/represented as the area of a rectangle. For example, consider the fact $2 \times 3 = 6$. Using repeated addition 2×3 means 2 groups of 3 or $3 + 3$.

Geometrically, this looks like the following:



2 and 3 are called factors of 6.
The factor 2 is the number of groups (rows).
The factor 3 is the number in each group (columns).

Notice that 6 square units cover the rectangle that is formed, so the <u>area</u> of the rectangle is 6.

This leads to a general definition:

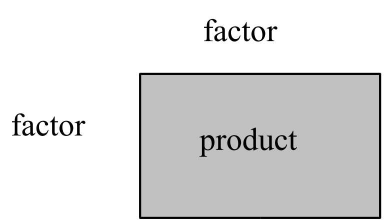
Definition: The area of a rectangle is the product of its length and width.

Moreover, the product of two numbers can be represented by this area.

i.e. area of rectangle = (length) • (width)

$\downarrow \qquad \downarrow \qquad \downarrow$

Product = (factor) • (factor)



Benefits of Using the Area Model

- Puts multiplication (and division) of whole numbers, fractions, decimals, and polynomials in a geometric context and reinforces the concept of area.
- Gives a visual/conceptual setting for the commutative property of multiplication and the distributive property which are fundamental to the real number system, multiplication algorithms, and manipulating algebraic symbols.
- Helps to explain why the “standard” algorithms for multiplication work.
- Gives an efficient, alternate way to record multiplying whole numbers as well as multiplying, factoring, and dividing polynomials.
- Connects arithmetic of whole numbers to algebra and algebraic thinking.
- Establishes the groundwork for helping visual learners in the conceptual understanding of multiplying, factoring, and dividing polynomials.

Examples of the Area Model Across the Curriculum

- Multiplication of whole numbers (Grades 2-5)
- Division of whole numbers (Grades 2-5)
- Number Theory – Factors & Primes (Grades 4-6)
- Fraction models – (Grades 3-6)
- Multiplication of fractions (Grades 5-6)
- Multiplication of decimals (Grades 5-6)
- Multiplication of polynomials (Grades 7-12)
- Factoring polynomials (Grades 7-12)
- Completing the Square (Grades 7-12)
- Division of polynomials (Grades 7-12)

Factoring Whole Numbers

What other possible rectangles can be formed with 6 square units? Use square tiles to figure this out, and then draw the rectangles on the grid below.

What are the dimensions (length and width) of the rectangles?

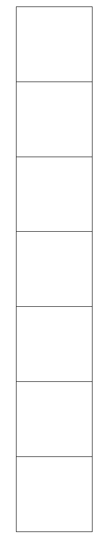
____

What are the factors of 6?___________________

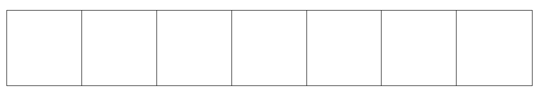
Do you notice any similarities between two of the rectangles?

What possible rectangles can be formed with 7 square units? Show in the grid above. How is this different than with 6 square units?

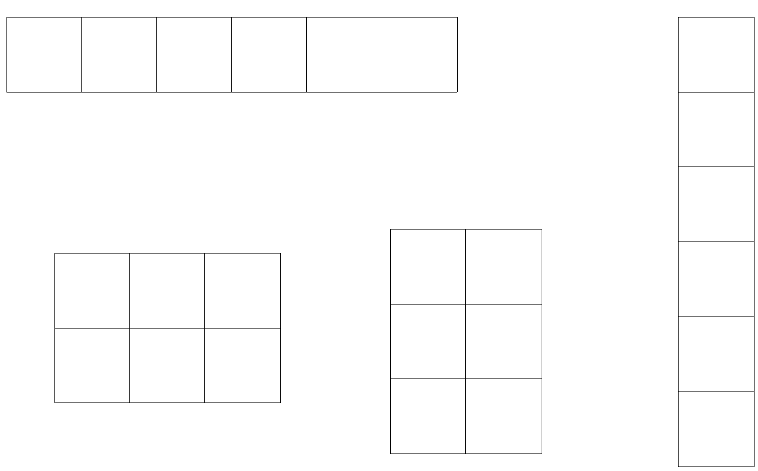
Factoring Whole Numbers



7 is a prime number because it has exactly two distinct factors, 1 and 7. These numbers correspond to the whole number dimensions of the rectangles that can be made with an area of 7 square units. Only two rectangles can be made.



On the other hand, 6 is not a prime number because more than two rectangles can be made with an area of 6 square units using whole number dimensions. As can be seen below, the possible dimensions (i.e. factors) are 1, 2, 3, and 6.



CLASS ACTIVITY: Discovering Factors and Primes with Square Tiles

Use square tiles to form all possible rectangular arrays that can be constructed for the numbers from 1 through 25 (remember: a square is a rectangle). Fill in the following table:

Number	Dimensions Of Rectangles	Number Of Rectangles	Factors	Number Of Factors	Prime Or Composite	Square Number?
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						
14						
15						
16						
17						
18						

19						
20						
21						
22						
23						
24						
25						

1. Which number can be made with only 1 rectangle?

2. Which numbers can be made with exactly 2 rectangles?
   These numbers are called **prime.**

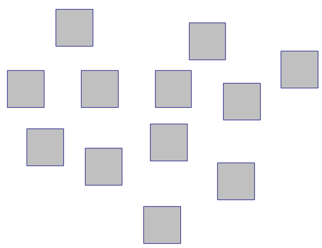
3. Which numbers can be made with more than 2 rectangles?
   These numbers are called **composite.**

4. Which numbers from 1 to 25 do you think could be called square numbers?
   Why? How many factors do these numbers have?

Division and Area

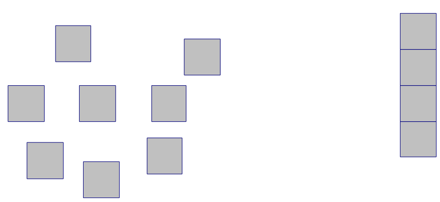
Division facts can also be illustrated using the areas of rectangles.
Here the dividend represents the area of the rectangle, the divisor is the given length of one of the sides, and the quotient is the length of the other side of the rectangle.

Example: Use 12 unit squares to find the quotient for $12 \div 4$.



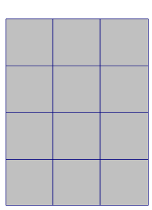
Using the notion of division as repeated subtraction, we will subtract groups of 4 squares from our 12 squares. We will arrange the squares as columns and try to build a rectangle with one side of length 4.

1. Remove the first group of 4 and arrange in a column.



2. Remove the second group of 4 and arrange in a column.

3. There is only one more group of 4 to remove. This gives a 4 x 3 rectangle.

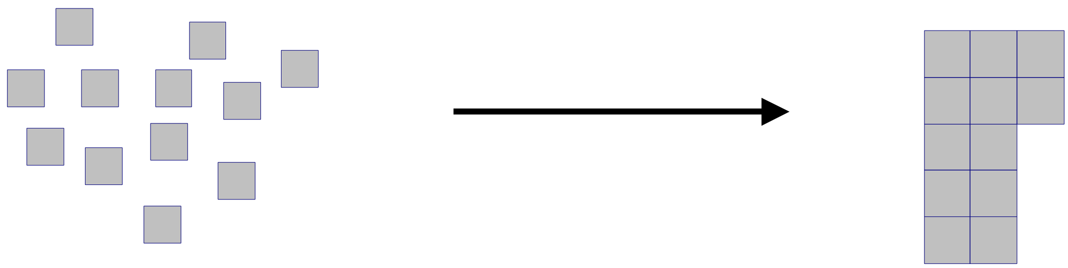


$12 \div 4 = 3$ because we were able to create 3 groups of 4. But if we consider the area of the rectangle, we see that $4 \times 3 = 12$. We have found the "missing factor" from the equation $4 \times \_\_ = 12$. Building the rectangle connects the repeated subtraction concept of division to the missing factor concept of division.

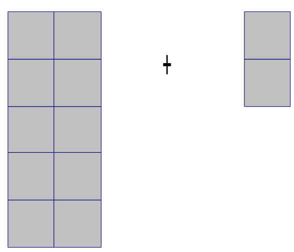
Division with remainders:

Now try this idea on the following division problem: $12 \div 5$

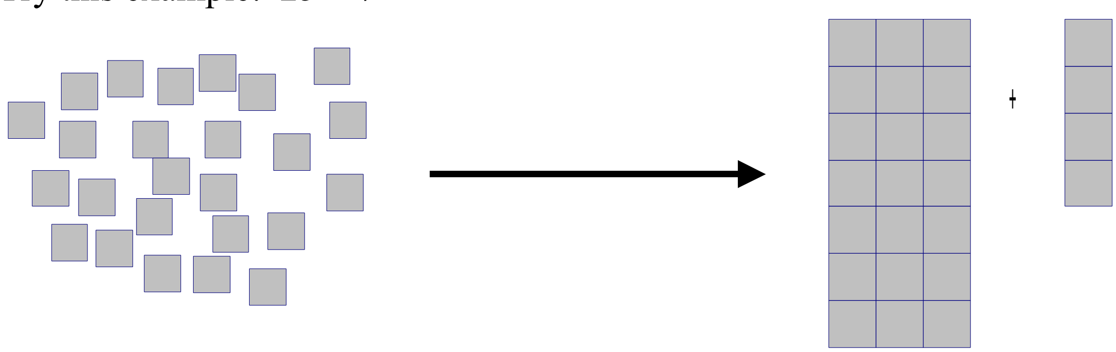
We will try to arrange our group of 12 squares into a rectangle with one side equal to 5 units:



We are unable to make a complete rectangle. We have a 5 x 2 rectangle with 2 squares left over. We say that $12 \div 5 = 2$ with a remainder of 2. Since the 2 left over are 2 parts of one of the groups of 5, we also say that $12 \div 5 = 2 + \frac{2}{5}$ or $2\frac{2}{5}$



Try this example: $25 \div 7$



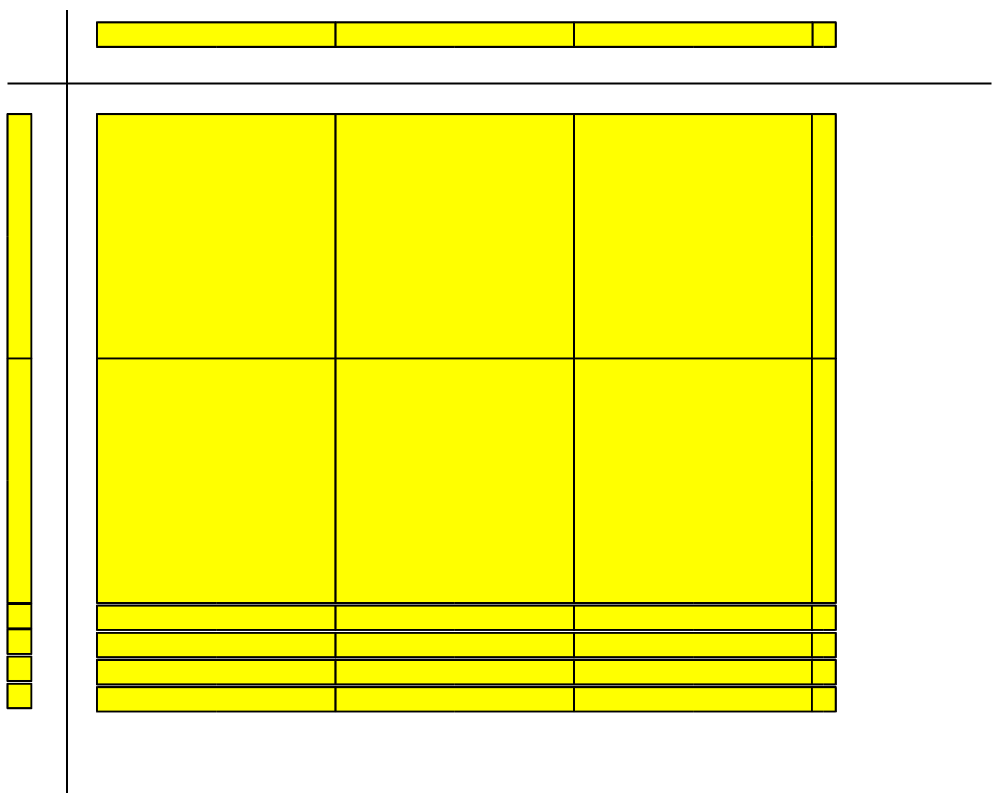
$25 \div 7 = 3$ with remainder 4

or

$25 \div 7 = 3\frac{4}{7}$

Multiplying Whole Numbers

The product of 31 and 24 is the area of the rectangle shown below created with base-ten blocks that has a width of 24 and a length of 31.



Therefore, $31 \times 24 = 600 + 20 + 120 + 4$

The corresponding "generic rectangle" is:

	30	1
20	600	20
4	120	4

Multiplying Whole Numbers

Use the area model to find 14×23 on the grid below.

Create the corresponding "generic rectangle".

Two-Digit Multiplication and Area with Base-10 Blocks

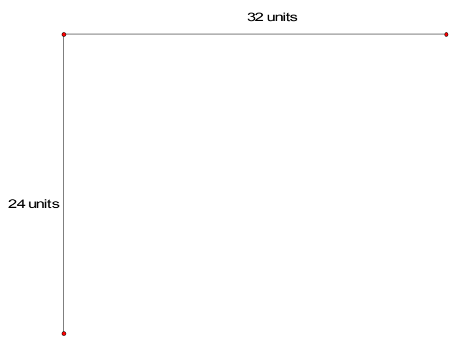
We have seen that we can compute a product **a x b** by finding the area rectangle with sides of length **a** and **b**. We can now extend this idea to two-digit numbers.

Example: In order to find 24×32 we will build a rectangle with sides of length 32 units and 14 units using Base-10 blocks.

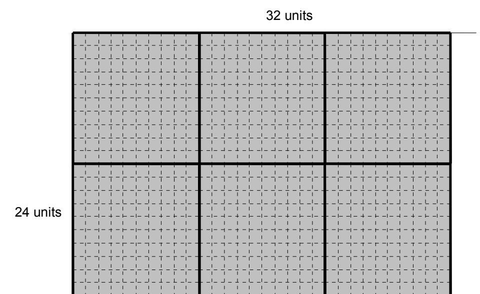
Considering only the top faces of the blocks,

- a base-10 unit has area 1unit x 1unit = 1 square unit
- a base-10 bar has area 10units x 1 unit = 10 square units
- a base-10 flat has area 10units x 10units = 100 square units.

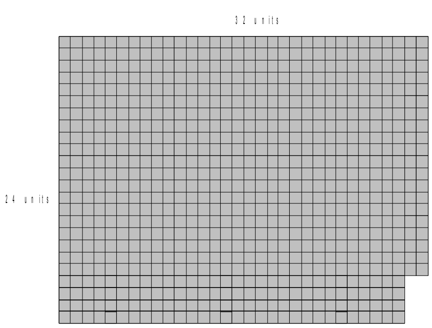
1. Build the frame:



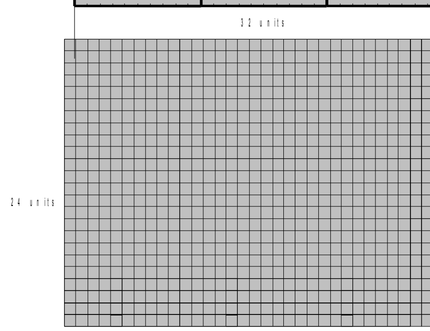
2. Starting with the left corner, fill in with as many flats as possible.



3. Now fill in as many bars as possible:



4. Complete the rectangle by filling in the lower right corner with unit rectangles:



5. The total area is 6 flats + 12 bars on the bottom + 4 bars on the side + 8 units = (6 x 100) + (12 x 10) +(4 x 10) + 8 = 600 + 120 + 40 + 8 = 768 square units.

So, **24 x 32 = 768**.
The area diagram gives us a way to understand the standard multiplication algorithm:

32

x 24

← 8 Multiply 4 x 2 (these are the 8 units)

12 ← 0 Multiply 4 x 30 (these are the 12 bars on the bottom)

4 ← 0 Multiply 20 x 2 (these are the 8 bars on the side)

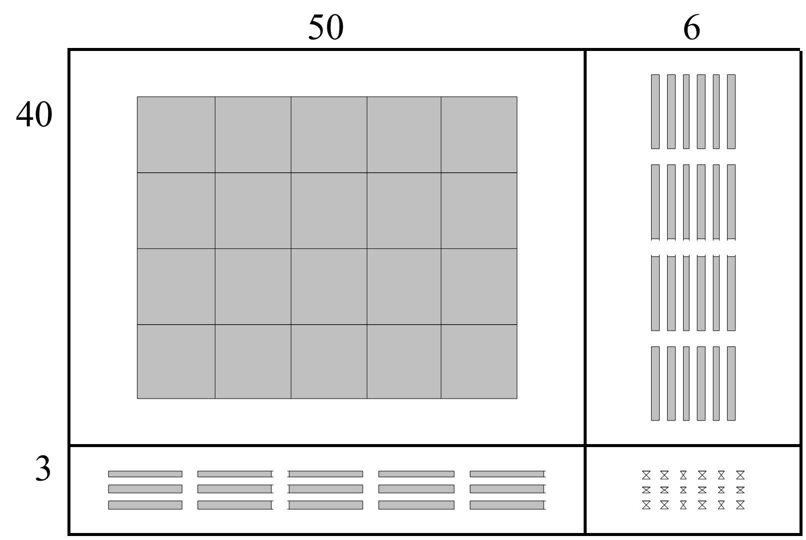
\+ ← 600 Multiply 20 x 30 (these are the six flats)

76 8

From Concrete to Abstract: Box Puzzles and the Distributive Property

Instead of actually building a rectangle, we could set up the areas as a box puzzle.

Example: 43 x 56

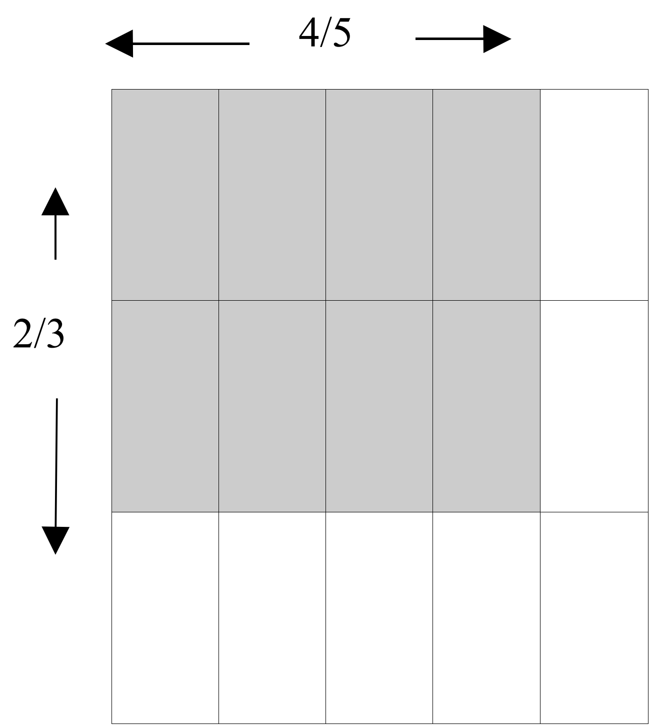


Now, we replace the pictures with their values in a "generic rectangle":

	50	6	56
40	2000	240	2240
3	150	18	168
43	2150	258	2408

Areas and Multiplying Fractions

The shaded rectangle with dimensions $\frac{2}{3}$ by $\frac{4}{5}$ has an area of $\frac{8}{15}$ of the unit square.

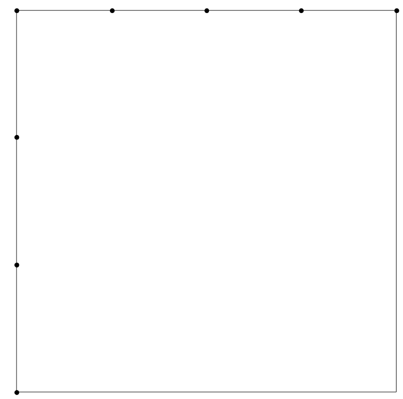


Therefore, $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$

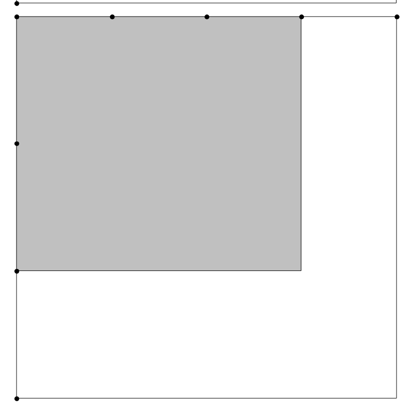
Areas and Multiplying Fractions

<u>Example</u>: Find the area of a rectangle that has one side equal to $\frac{2}{3}$ of a unit and the other side equal to $\frac{3}{4}$ of a unit.

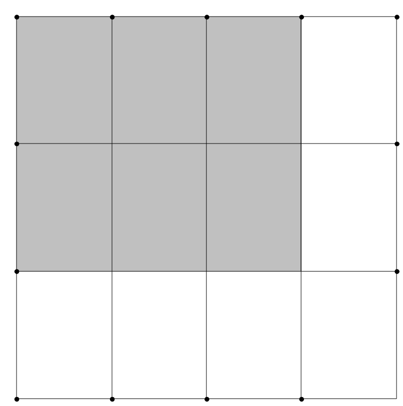
1) Start with the unit square. Divide the left side into three equal parts and the top into four equal parts.



2) Shade in the area that is $\frac{2}{3} \times \frac{3}{4}$



3) We can visualize the product if we partition the square into rectangles using the division marks on the sides.



Here we see that the unit square is divided into 12 congruent rectangles, of which 6 are shaded.

So we see that $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$

Note: Six of the smaller rectangles are shaded and six are unshaded, so exactly one-half of the unit square is shaded. Therefore $\frac{6}{12}$ is equivalent to $\frac{1}{2}$.

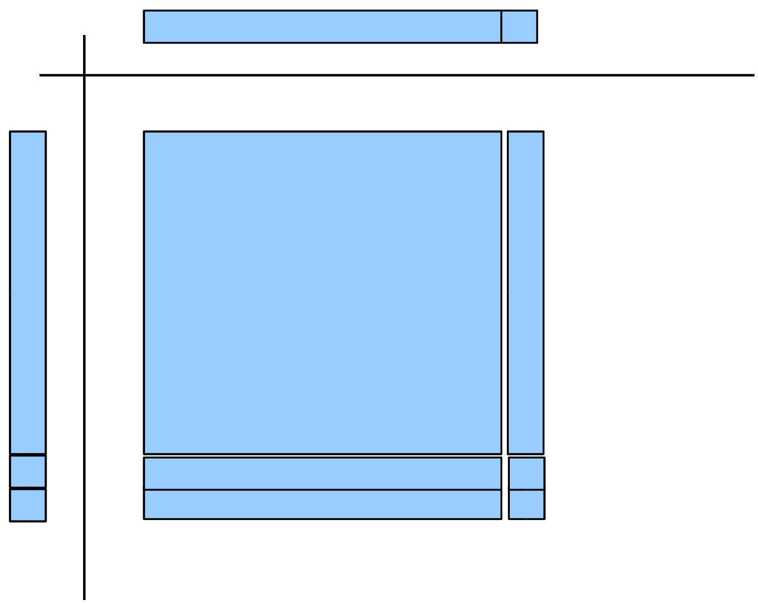
Use the Area Model of Multiplication to find each product:

1) $\frac{1}{3} \times \frac{1}{6}$

2) $\frac{1}{4} \times \frac{3}{2}$

Multiplying Polynomials

Using algebra tiles (the long tile is $+x$, the small square tile is $+1$, the large square tile is $+x^2$) the product of $x+1$ and $x+2$ is the area of the shaded rectangle shown below.



.

Therefore, $(x+1)(x+2) = x^2 + 3x + 2$.

The corresponding generic rectangle is:

	x	2
x	x^2	$2x$
1	x	1

Note that the "like terms" appear diagonally.

Use Algebra Tiles:

1) $(2x+1)(x-3)$ 2) $(x+2)^2$

Use "generic rectangles"

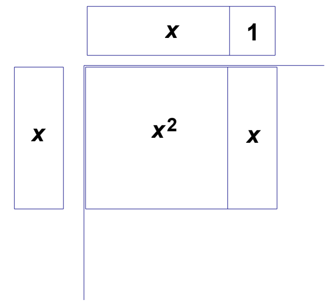
1) $(2x+1)(x-3)$ 2) $(x+2)^2$

3) $(x^2-x+1)^2$

Factoring Polynomials with Algebra Tiles

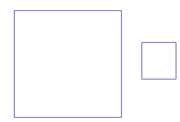
A polynomial is *factored* if it can be expressed as a product of polynomials.

For example, $x^2 + x = x(x+1)$.

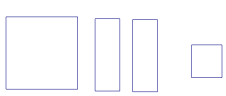


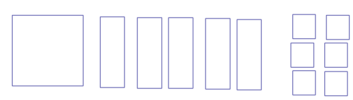
Using algebra tiles and the area model of multiplication, this means that we can create a rectangle whose area is $x^2 + x$, length is x, and width is $x+1$ See figure at the right.

If a polynomial cannot be factored (other than 1 and itself), then the polynomial is said to be "prime". For example, the tiles for $x^2 + 1$ cannot be arranged in a rectangle, so $x^2 + 1$ is not factorable



For the following examples, try to arrange tiles into rectangles. Then find the length and width of each rectangle. Write the polynomial in factored form, if possible.





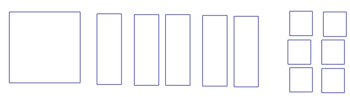
Modeling the Traditional Approach to Factoring Trinomials of the form ax^2+bx+c using Algebra Tiles:

- Lay out the appropriate number of tiles to represent ax^2+bx+c.
- Make a frame.
- Start with the x^2 tiles : make a rectangle in the upper left corner of the frame using the x^2 tiles.
- Then use the unit tiles: make a rectangle in the lower right corner of the frame using the <u>unit tiles</u>.
  <u>Note:</u> You may be able to make more than one-sized rectangle for each corner. It depends on how many x tiles you have to work with – see next step.

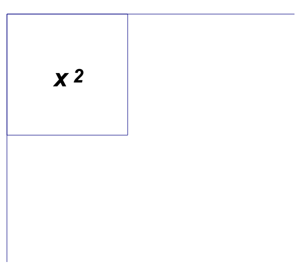
- Fill in the remaining two corners with the given x tiles (all the tiles in each corner must be the same color). You may need to add zero pairs to do this.
  Don't forget: $+\cdot+=+$, $+\cdot-=-$, $-\cdot+=-$, and $-\cdot-=+$

- Frame the rectangle. The two polynomials outside the rectangle are the factors of the polynomial.

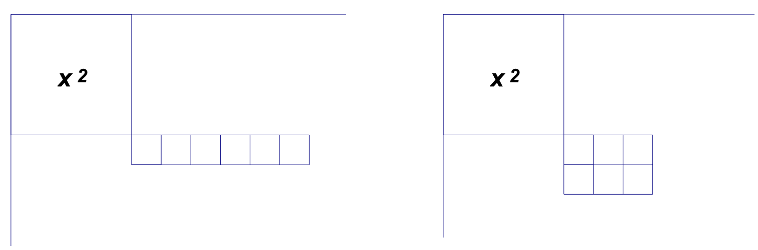
Example: x^2+5x+6



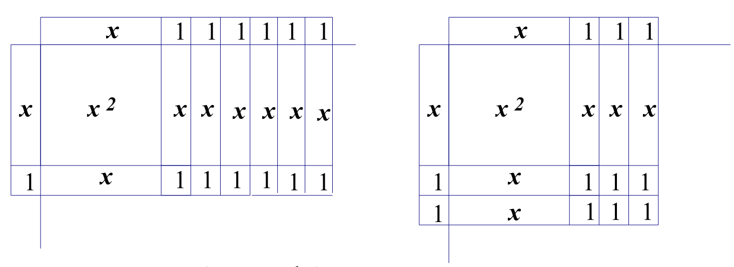
1) Make frame and lay x^2 tile



2) Make all possible arrangements of unit tiles .



3) Fill in corners with 5 x tiles.
Note the first arrangement uses 7 x tiles (too many!).



Doesn't work!

This is correct!
So $x^2+5x+6=(x+2)(x+3)$.

Dividing Polynomials with Algebra Tiles

According to the *division algorithm*,

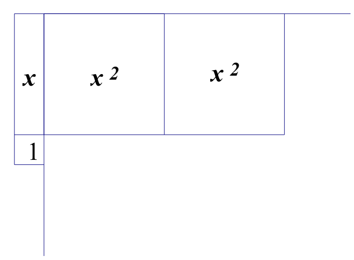
Dividend = (Divisor)(Quotient) + Remainder

So when using the Area Model for a division problem, we use the tiles representing the divisor as a given length of one of the sides of a rectangle and use the tiles representing the dividend as the area of the rectangle (using as many tiles as possible). The goal is to find the length of the other side of the rectangle . Extra tiles from the dividend become part of the remainder.

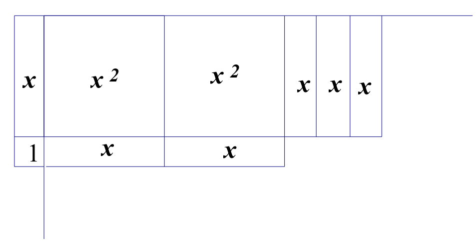
Note: It may be necessary to add zero pairs to create the rectangle.

<u>Example</u>: $(2x^2 + 5x + 3) \div (x + 1)$

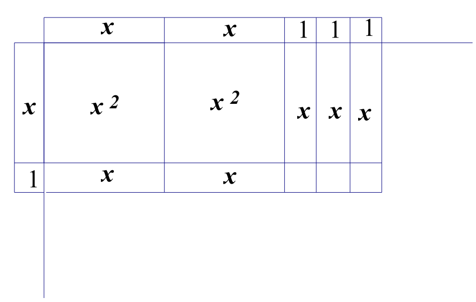
Start by placing $x + 1$ on the outside of the rectangle and $2x^2$ inside the rectangle in the upper left corner as shown.



Place 2 of the 5 given *x* tiles in the lower left corner and 3 *x* tiles in the upper right corner.



Place the 3 given units in the lower right corner.

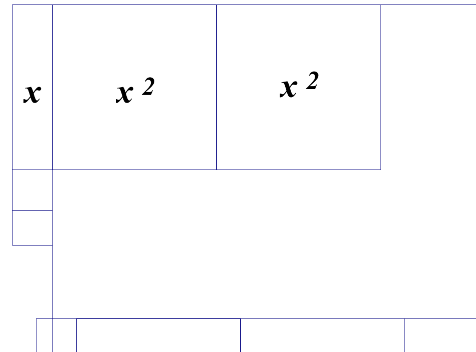


All tiles have been used, so the remainder is 0. The quotient is $2x + 3$ as shown at the top of the rectangle.

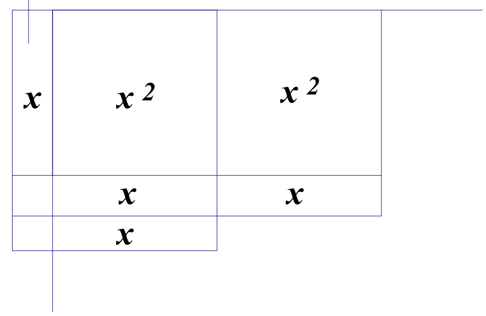
<u>Note</u>: You can use the picture to check using multiplication!

<u>Example</u>: $(2x^2 + 3x - 5) \div (x + 2)$

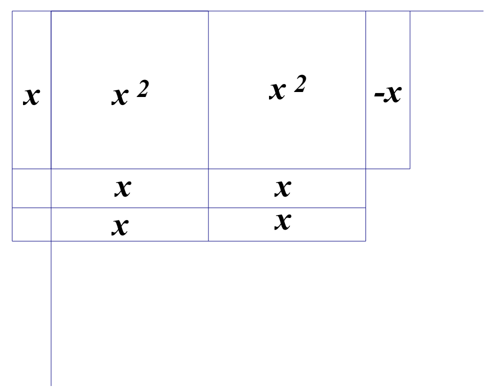
Start by placing $x+2$ on the outside of the rectangle and $2x^2$ inside the rectangle in the upper left corner as shown.



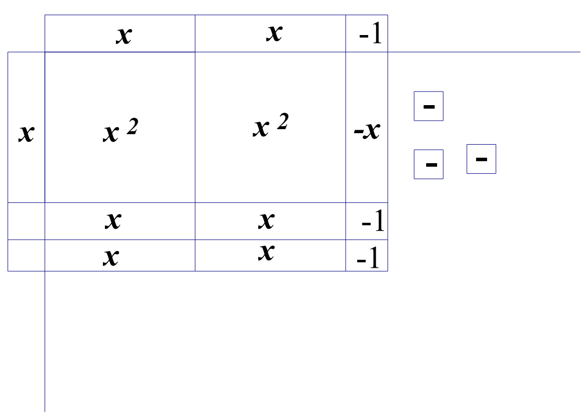
Place all 3 given x tiles in the lower left corner. To fill in this corner we need to add a zero pair (one positive/one negative tile) and place the positive in lower left and negative in upper right.



Now place 2 of the 5 given negative unit tiles in the lower right corner. There will be 3 negative tiles left over.



Therefore, the remainder is -3 and the quotient is $2x-1$



Use the picture to check!

Dividing Polynomials with Generic Rectangles

Generic rectangles can be used to divide polynomials of any degree. Use the divisor to the left of the frame and the dividend inside the frame (similar to long division) to form a rectangle with the Area Model in mind.

<u>Example</u>: Find the quotient and remainder: $(x^3+7x^2+8x+5)\div(x+1)$
Place x^3 in the box in the upper left corner. Divide x^3 by x to get x^2. Place x^2 above the rectangle as shown.

	x^2			
x	x^3			
1				

Multiply x^2 by 1 to complete the first column of the rectangle. Remember that "like terms" a grouped diagonally.

	x^2	$6x$		
x	x^3	$6x^2$		
1	x^2			

Since $7x^2$ is in the dividend, place $6x^2$ diagonally across from x. Divide $6x^2$ by x and place in the quotient.

Continue to get the following:

Quotient: x^2+6x+2
Remainder: 3

	x^2	$6x$	2	
x	x^3	$6x^2$	$2x$	3
1	x^2	$6x$	2	

Use generic rectangles to find the quotient and remainder:

1) $(x^4 - 5x^3 + 3x - 4) \div (x - 2)$

2) $(4x^3 - 2x^2 + 2x - 3) \div (2x + 1)$

3) $(x^4 - x^3 - 2x^2 + x + 3) \div (x^2 + x + 1)$

References

- *Mathematics for Teachers, an Exploratory Approach to Arithmetic, Algebra, and Geometry,* Second Edition, by B. Stein with L. Wallace, Kendall Hunt Publishing Company, 2009.

- *Discovering Algebra, an Investigative Approach*, by J. Murdock, E. Kamischke, and E. Kamischke, Key Curriculum Press, 2008.

- Foam algebra tiles and other manipulatives available at http://www.thinkitbyhand.com/.

- Algebra tile masters (paper versions) and a complete set of handouts are available online at the MaTHink 2010 website http://algebraforum.wikispaces.com/ under Laura Wallace's link.